# Coverings That Preserve Sense of Direction*

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## Abstract

Sense of direction is a property of labelled networks (i.e., arc-coloured graphs) that allows one to assign coherently local identifiers to other processors on the basis of the route followed by incoming messages. We prove that (weak) sense of direction is preserved by the construction of regular coverings (i.e., coverings induced by voltage assignments in a group) whose voltage assignment depends only on colours. Moreover, this construction preserves minimality.

**Key words:** distributed systems, interconnection networks, sense of direction, graph coverings.

## 1 Introduction

*Sense of direction* [7] is a property of global consistency of the colouring of a network that can be used to reduce the complexity of distributed algorithms [6]. Although there are polynomial algorithms for checking whether a given coloured graph has (weak) sense of direction [4], the polynomial bounds are rather high, and, moreover, there are no results (besides the obvious membership to NP) about *finding* a colouring that is a (weak) sense of direction using a given (or smallest) number of colours.

In this note we show that a standard graph-theoretical construction (the regular covering induced by a voltage assignment in a group) preserves (weak) sense of direction under the assumption that the voltage assignment depends only on the colours. Moreover, the number of colours used is not increased, although the construction multiplies the number of arcs and nodes of the base graph by the cardinality of the group. This property is of particular interest because several classes of networks in the literature are coverings of this kind (in the last section, we work out the examples of butterflies and extended cube-connected cycles). Thus, it is possible to give (weak) sense of direction to these large networks just using the colours that are necessary for a small base graph.

## 2 Sense of direction

We follow the notation of [4], with slight extensions to handle parallel arcs. A *(directed) (multi)graph* $G$ is defined by a nonempty set $N_G$ of nodes, a set $A_G$ of arcs, and by two functions $s_G, t_G : A_G \to N_G$ specifying the source and the target of each arc (the subscript $G$ will be omitted whenever no confusion is possible). We write $P[x, y] \subseteq A^*$ for the set of paths from the node $x$ to the node $y$.

A *colouring* of a graph $G$ is a function $\lambda : A \to \mathscr{L}$, where $\mathscr{L}$ is a finite set of colours; the map $\lambda^* : A^* \to \mathscr{L}^*$ is defined by $\lambda^*(a_1 a_2 \cdots a_p) = \lambda(a_1)\lambda(a_2)\cdots\lambda(a_p)$. Given a graph $G$ coloured by $\lambda$, let

$$L(x, y) = \{\lambda^*(\pi) \mid \pi \in P[x, y]\};$$

in other words, $L(x, y)$ is the language recognized by the automaton with transition graph $G$ when $x$ is the initial state and $y$ is the final state; moreover, $L = \bigcup_{\langle x,y\rangle \in N^2} L(x, y)$ is the set of all strings of colours appearing in $G$.

A *local naming* for $G$ is a family of injective functions $\beta = \{\beta_x : N \to \mathscr{S}\}_{x\in N}$, with $\mathscr{S}$ a finite set, called the

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*name space*. Intuitively, each node $x$ of $G$ gives to each other node $y$ a name $\beta_x(y)$ taken from the name space.

Given a coloured graph endowed with a local naming, a function $f : L \to \mathscr{S}$ is a *coding function* iff

$$\forall x, y \in N \quad \forall \pi \in P[x, y] \qquad f(\lambda^*(\pi)) = \beta_x(y).$$

A coding function translates the colouring of the path along which two nodes $x$, $y$ are connected into the name that $x$ gives to $y$. A colouring $\lambda$ is a *weak sense of direction* for a graph $G$ iff for some local naming there is a coding function. We shall also say that a coloured graph *has* weak sense of direction, or that $\lambda$ *gives* weak sense of direction to $G$.

As an example, consider a $p \times q$ torus where the links have the standard *compass colouring* (North/South/East/West). Each node with coordinates $\langle i, j\rangle$ (where $i \in \mathbf{Z}_p$, $j \in \mathbf{Z}_q$) gives to a node with coordinates $\langle h, k\rangle$ the local name $\langle i - h, j - k\rangle$. If a message arrives through a path coloured by $v$, the receiver knows the sender under the local name $f(v) = \langle \#_N(v) - \#_S(v), \#_E(v) - \#_W(v)\rangle$, where $\#_N(v)$ is the number of occurrences of the "North" colour in $v$, and so on. On the other hand, if we reverse the North/South colouring at just one node, then the resulting graph is not a weak sense of direction, because there is no global way to detect whether the string "North North North" corresponds to a first-coordinate offset of 1 or 3.

Finally, a *decoding function* is a map $d : \mathscr{L} \times \mathscr{S} \to \mathscr{S}$ that satisfies

$$\forall a \in A \quad \forall x \in N \quad \forall \pi \in P[t(a), x] \qquad d(\lambda(a),\ f(\lambda^*(\pi))) = \beta_{s(a)}(x).$$

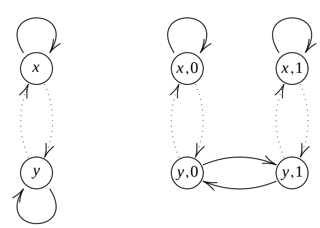
A decoding function translates the name given by $t(a)$ to $x$ into the name given by $s(a)$ to $x$ knowing only the colour of the arc $a$. A colouring $\lambda$ is a *sense of direction* for a graph $G$ iff for some local naming there is a coding function and a decoding function. It is easy to check that the previous example is a sense of direction, too: the decoding function modifies the local name by adding the contribution of the colour (e.g., it increments the first coordinate by one if the colour is "North").

# 3 Regular coverings

**Definition 1** Let $G$ be a graph, $\Gamma$ a group and $v : A \to \Gamma$ a function, called a *voltage assignment*. The *regular covering* of $G$ with respect to $v$ is the graph $G^v$ having set of nodes $N \times \Gamma$ and set of arcs $A \times \Gamma$, with $s(\langle a, g\rangle) = \langle s(a), g\rangle$ and $t(\langle a, g\rangle) = \langle t(a), gv(a)\rangle$.

The definition above formalizes the intuition of a large graph that is "locally like" a small graph and has a high degree of internal symmetry. The construction of regular coverings by means of voltage assignments is well known in topology and (undirected) graph theory (see, for instance, [2, 10]). Definition 1 is just the obvious generalizations to (directed multi)graphs of the standard voltage graph construction introduced by Gross [9]. There is also a geometric counterpart to the construction, namely, a covering projection from $G^v$ to $G$, which will not be used in this note.

The regular covering of a coloured graph has an obvious induced colouring (just colour $\langle a, g\rangle$ like $a$). However, regular coverings do not preserve sense of direction, as we can see in the following example (we use continuous and dotted patterns to represent colours):




The graph on the right clearly does not have weak sense of direction (just check the different lengths of the continuous cycles). Yet, it is obtained from the left-hand graph (which has sense of direction) by the voltage assignment in $\mathbf{Z}_2$ which is nonzero only on the loop at $y$.

We need some further restriction to guarantee that the construction is "uniform" with respect to the given colouring, and this intuitive idea is formalized as follows:

**Definition 2** A voltage assignment $v$ on a coloured graph $G$ is *colour uniform* iff $v$ extends along $\lambda$, that is, $v(a) = v(b)$ when $\lambda(a) = \lambda(b)$.

Since in the colour-uniform case $v$ depends only on colours, with a slight abuse of notation we shall sometimes write $v(\sigma)$, with $\sigma \in \mathscr{L}$, instead of $v(a)$, where $a$ is any arc with colour $\sigma$. The main theorem of this note claims that colour-uniform regular coverings have (weak) sense of direction when the base graph has:

**Theorem 1** Let $\lambda$ be a (weak) sense of direction for $G$, and $v : A \to \Gamma$ a colour-uniform voltage assignment. Then $\lambda^v(\langle a, g\rangle) = \lambda(a)$ is a (weak) sense of direction for $G^v$.

**Proof.** Let $f$, $\mathscr{S}$ and $\beta$ be the coding functions, name space and local naming of $G$. We define a local naming $\beta^v$ for $G^v$ with name space $\mathscr{S} \times \Gamma$ as follows:

$$\beta^v_{\langle x,g\rangle}(\langle y, h\rangle) = \langle \beta_x(y), g^{-1}h\rangle.$$

Then,

$$f^v(\sigma_1\sigma_2\cdots\sigma_p) = \langle f(\sigma_1\sigma_2\cdots\sigma_p), v(\sigma_1)v(\sigma_2)\cdots v(\sigma_p)\rangle$$

is a coding function for $G^v$. Indeed, suppose $\langle a_1, g_1\rangle\langle a_2, g_2\rangle\cdots\langle a_p, g_p\rangle$ is a path from $\langle x, g_1\rangle$ to $\langle y, g_{p+1}\rangle$. Then necessarily $g_{i+1} = g_i v(a_i)$ for $i = 1, 2, \ldots, p$, so

$$\begin{aligned} &f^v((\lambda^v)^*(\langle a_1, g_1\rangle\langle a_2, g_2\rangle\cdots\langle a_p, g_p\rangle)) \\ &\quad= f^v(\lambda^*(a_1a_2\cdots a_p)) \\ &\quad= \langle f(\lambda^*(a_1a_2\cdots a_p)), v(a_1)v(a_2)\cdots v(a_p)\rangle \\ &\quad= \langle \beta_x(y), g_1^{-1}g_2g_2^{-1}g_3\cdots g_p^{-1}g_{p+1}\rangle \\ &\quad= \langle \beta_x(y), g_1^{-1}g_{p+1}\rangle \\ &\quad= \beta^v_{\langle x,g_1\rangle}(\langle y, g_{p+1}\rangle). \end{aligned}$$

If we have also a decoding function $d$ for $G$, then a decoding function $d^v$ for $G^v$ can be defined as follows:

$$d^v(\sigma, \langle \zeta, g\rangle) = \langle d(\sigma, \zeta), v(\sigma)g\rangle,$$

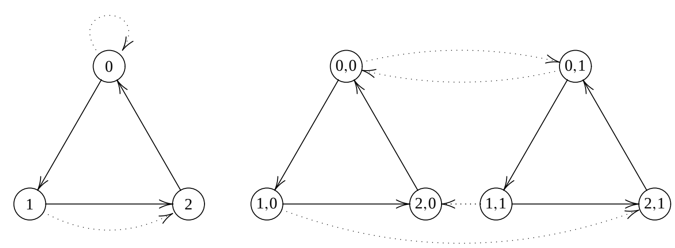
where $\sigma \in \mathscr{L}$ and $\zeta \in \mathscr{S}$. Indeed, suppose $\langle a, g\rangle$ is an arc from $\langle x, g\rangle$ to $\langle y, h\rangle$ (where $h = gv(a)$), and that $\pi$ is a path from $\langle y, h\rangle$ to $\langle z, k\rangle$. Then

$$\begin{aligned} &d^v(\lambda^v(\langle a, g\rangle),\ f^v((\lambda^v)^*(\pi))) \\ &\quad= d^v(\lambda^v(\langle a, g\rangle), \beta^v_{\langle y,h\rangle}(\langle z, k\rangle)) \\ &\quad= d^v(\lambda(a), \langle \beta_y(z), h^{-1}k\rangle) \\ &\quad= \langle d(\lambda(a), \beta_y(z)), v(a)h^{-1}k\rangle \\ &\quad= \langle \beta_x(z), v(a)v(a)^{-1}g^{-1}k\rangle \\ &\quad= \langle \beta_x(z), g^{-1}k\rangle \\ &\quad= \beta^v_{\langle x,g\rangle}(\langle z, k\rangle). \ \blacksquare \end{aligned}$$

The converse of the previous theorem could be phrased as follows:

> Let $\lambda$ be a colouring of $G$, and $v : A \to \Gamma$ a colour-uniform voltage assignment. If $\lambda^v(\langle a, g\rangle) = \lambda(a)$ is a (weak) sense of direction for $G^v$, then $\lambda$ is a (weak) sense of direction for $G$.

Unfortunately, the above statement is false (even assuming sense of direction on $G^v$ and requiring only weak sense of direction on $G$), as shown by the following counterexample:




$G$ $\qquad$ $G^v$

The graph $G$ does not have weak sense of direction, since one of the colours is associated both to a loop and to a nonloop. Yet, $G^v$ (where $v : A \to \mathbf{Z}_2$ is the colour-uniform voltage assignment associating 0 to continuous arcs, and 1 to dotted arcs) has sense of direction.

The verification of the latter claim is by no means obvious, and has been performed mechanically by implementing the algorithms described in [4]. The resulting local naming $\beta$ has name space $\mathbf{Z}_3 \times \mathbf{Z}_2$, and satisfies the following equation:

$$\beta_{\langle x,a\rangle}(\langle y, b\rangle) = \langle y + [(a \neq b) - (a = b)]x, a + b\rangle,$$

where $x, y \in \mathbf{Z}_3$, $a, b \in \mathbf{Z}_2$, and we used Iverson's notation [8] (a formula enclosed in parenthesis takes value 1 when it is true, 0 otherwise). The coding function can be easily obtained from $\beta$, and the decoding function $d$ is given by

$$d(\text{——}, \langle x, a\rangle) = \langle x + a + 1, a\rangle$$
$$d(\cdots\cdots, \langle x, a\rangle) = \langle x, a + 1\rangle,$$

where, with a slight abuse of notation, we are adding $a$ to $x$, even if they live in different groups.

The *Cayley graph* of a group $\Gamma$ with respect to a set (or multiset) of elements $S = \{g_1, g_2, \ldots, g_p\} \subseteq \Gamma$ has node set $\Gamma$, arc set $\Gamma \times S$, $s(\langle g, g_i\rangle) = g$ and $t(\langle g, g_i\rangle) = gg_i$. Theorem 1 generalizes the known results about the sense of direction of Cayley graphs, which have a natural colouring in $S$ given by $\lambda(\langle g, g_i\rangle) = g_i$ (see, e.g., [5]), to colour-uniform regular coverings: in the general case, (weak) sense of direction is guaranteed *locally* by the structure of $G$, and *globally* by the structure of the voltage group $\Gamma$. Indeed, since Cayley graphs are regular coverings of bouquets (graphs with one node), a trivial consequence of Theorem 1 is that they can be given a sense of direction (of course, expressing a Cayley graph as a regular covering of a bouquet means constructing a voltage assignment that allows one to derive immediately the natural colouring of the graph).

A final comment should be made about minimality. A graph has *minimal* (weak) sense of direction if it can be given (weak) sense of direction using as many colours as its maximum outdegree. In [3] we showed that outregular graphs with a minimal (weak) sense of direction are exactly the Cayley graphs. For graphs that are not outregular, it is easy to show that minimality is preserved (and, in fact, also reflected) by colour-uniform regular coverings:

**Corollary 1** $G$ has minimal (weak) sense of direction iff $G$ is the colour-uniform regular covering of a graph $H$ with minimal (weak) sense of direction.

**Proof.** The "only if" part is trivial (take $G = H$). In view of Theorem 1, the other direction requires just to show that $H^\upsilon$ has the same maximum outdegree as $H$, which is easily checkable from the definition of regular covering. ∎

## 4 Strongly connected coverings

Strongly connected graphs may give rise to disconnected coverings; since one is usually interested in networks that are strongly connected, we give some results characterizing, on the basis of the languages $L(x, y)$, those voltage assignments $\upsilon$ that produce strongly connected coverings. We denote with $\hat{\upsilon}$ the map $\mathscr{L}^* \to \Gamma$ induced by the free monoid property, that is, $\hat{\upsilon}(\sigma_1\sigma_2\cdots\sigma_p) = \upsilon(\sigma_1)\upsilon(\sigma_2)\cdots\upsilon(\sigma_p)$, and silently extend $\hat{\upsilon}$ to the direct image map $2^{\mathscr{L}^*} \to 2^\Gamma$.

**Theorem 2** Let $G$ be strongly connected. Then $G^\upsilon$ is strongly connected iff $\hat{\upsilon}(L(x, y)) = \Gamma$ for all pairs $x, y$ of nodes of $G$.

**Proof.** Assume $G^\upsilon$ is strongly connected. Since

$$L(\langle x, g\rangle, \langle y, h\rangle) = L(x, y) \cap \{w \in \mathscr{L}^* \mid h = g\hat{\upsilon}(w)\},$$

if we fix $x$ and $y$ and let $g = 1$, $L(\langle x, 1\rangle, \langle y, h\rangle) \neq \varnothing$ for all $h$ implies that for each $h$ there is a path $\pi$ from $x$ to $y$ in $G$ such that $\hat{\upsilon}(\pi) = h$, so $\hat{\upsilon}(L(x, y)) = \Gamma$.

For the converse, we know that for every pair of elements $g$, $h$ of $\Gamma$ and for every pair of nodes $x$, $y$ of $G$ there is a string $w \in L(x, y)$ such that $\hat{\upsilon}(w) = g^{-1}h$. Let $\pi = a_1a_2\cdots a_p$ be the path whose colour string is $w$. Then the path

$$\langle a_1, g\rangle\langle a_2, g\upsilon(a_1)\rangle \cdots \langle a_p, g\upsilon(a_1)\upsilon(a_2)\cdots\upsilon(a_{p-1})\rangle$$

starts from $\langle x, g\rangle$ and ends at

$$\langle y, g\upsilon(a_1)\upsilon(a_2)\cdots\upsilon(a_p)\rangle = \langle y, g\hat{\upsilon}(w)\rangle = \langle y, gg^{-1}h\rangle = \langle y, h\rangle. ∎$$

We remark that the conditions of the previous theorem are decidable, because $L(x, y)$ is regular. Moreover, the $n^2$ equalities of the statement can, in fact, be reduced to just $n$:

**Corollary 2** Let $G$ be strongly connected. Then $G^v$ is strongly connected iff for all nodes $x$ of $G$ we have $\hat{v}(L(x, x)) = \Gamma$.

**Proof.** Let $x, y$ be nodes of $G$ and $\pi$ a path from $x$ to $y$. Then, by hypothesis, when $\rho$ ranges over $P[x, x]$ (i.e., $\rho$ is an oriented cycle passing through $x$) $\hat{v}(\rho)$ covers all of $\Gamma$, and consequently the same happens for $\hat{v}(\rho\pi) = \hat{v}(\rho)\hat{v}(\pi)$. ∎

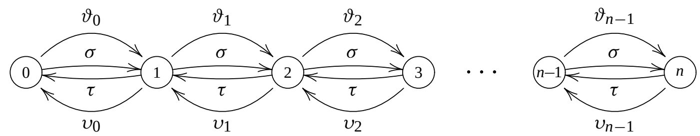
Alternatively, one just needs to check that the conditions of Theorem 2 are true along the arcs of an arbitrary spanning subgraph of $G$ (a *spanning subgraph* of a graph $G$ is defined by a subset $H$ of the arcs of $G$ that covers all of its nodes, i.e., for all nodes $x$ of $G$ we have $x = s(a)$ or $x = t(a)$ for some arc $a$ of $H$), so we may further reduce the number of equalities to be checked to $n/2$ if $G$ has a perfect matching:

**Corollary 3** Let $G$ be strongly connected, and $H$ a spanning subgraph of $G$. Then $G^v$ is strongly connected iff for all arcs $a$ of $H$ we have $\hat{v}(L(s(a), t(a))) = \Gamma$.

**Proof.** By Corollary 4, we just need to show that for all nodes $x$ of $G$ we have $\hat{v}(L(x, x)) = \Gamma$, and we prove only the case $x = s(a)$ for an arc $a$ of $H$. There is certainly a path $\pi$ (in $G$) going from $y = t(a)$ to $x$, which can be prefixed by path $\rho$ (again in $G$) going from $x$ to $y$. When $\rho$ varies through $P[x, y]$, $\hat{v}(\rho)$ varies through all of $\Gamma$, and consequently the same happens for $\hat{v}(\rho\pi) \subseteq \hat{v}(L(x, x))$. ∎

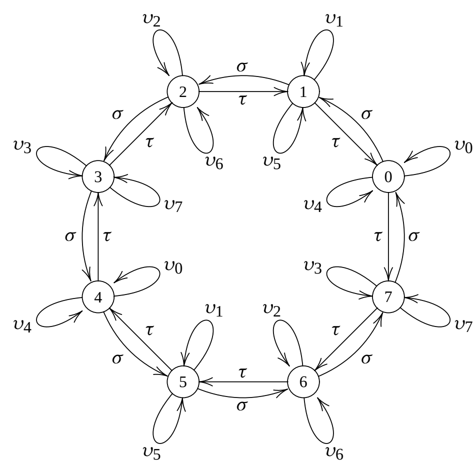
## 5 Applications

**Butterfly networks.** A *butterfly network* [12] with $2^n$ inputs is a graph with node set $\{0, 1, \dots, n\} \times (\mathbf{Z}_2)^n$, with an edge (i.e., a pair of opposite arcs) between the nodes $\langle l, r\rangle$, $\langle l+1, r\rangle$ and between the nodes $\langle l, r\rangle$, $\langle l+1, r+e_l\rangle$, for $0 \le l < n$; here $e_i$ is the element of $(\mathbf{Z}_2)^n$ that is zero everywhere, except for the $(i + 1)$-th component. It is not difficult to check that a butterfly with $2^n$ inputs is the colour-uniform regular covering of the following graph, with voltage assignment in $(\mathbf{Z}_2)^n$ given by $v(\sigma) = v(\tau) = 0$, $v(\vartheta_i) = v(\upsilon_i) = e_i$:




Thus, we can give sense of direction to a butterfly of $(n+1)2^n$ nodes using $2(n + 1)$ colours.

**(Extended) cube-connected cycles.** A (perfect) *extended cube-connected cycles (ECCC)* [1] with $2^r$-sized cycles has node set $\mathbf{Z}_{2^r} \times (\mathbf{Z}_2)^{2^r}$, with an edge between the node $\langle l, r\rangle$ and nodes $\langle l + 1, r\rangle$, $\langle l, r + e_l\rangle$ and $\langle l, r + e_{l+2^{r-1}}\rangle$. Again, it is not difficult to check that an ECCC is the colour-uniform regular covering of the following graph, with voltage assignment in $(\mathbf{Z}_2)^{2^r}$ (we show the case $r = 3$) given by $v(\sigma) = v(\vartheta) = 0$, $v(\upsilon_i) = e_i$:




Thus, we can give sense of direction to an ECCC of $2^{r+2^r}$ nodes using $2+2^r$ colours. Note that the same considerations hold for classical *cube-connected cycles* [11]—just delete the inner loops.

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