More Lower Bounds for Weak Sense of Direction: The Case of Regular Graphs

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Abstract A graph G with n vertices and maximum degree Δ_G cannot be given weak sense of direction using less than Δ_G colours. It is known that n colours are always sufficient, and it was conjectured that just $\Delta_G + 1$ are really needed, that is, one more colour is sufficient. Nonetheless, it has just been shown [2] that for sufficiently large n there are graphs requiring $\omega(n/\log n)$ more colours than Δ_G. In this paper, using recent results in asymptotic graph enumeration, we show not only that (somehow surprisingly) the same bound holds for regular graphs, but also that it can be improved to $\Omega(n \log\log n/\log n)$. We also show that $\Omega\left(d_G\sqrt{\log\log d_G}\right)$ colours are necessary, where d_G is the degree of G.

1 Introduction

Sense of direction and *weak sense of direction* [5] are properties of global consistency of the colouring of a network that can be used to reduce the complexity of many distributed algorithms [4]. Although there are polynomial algorithms for checking whether a given coloured graph has (weak) sense of direction [1], the polynomial bounds are rather high, and, moreover, there are no results (besides the obvious membership to NP) about finding a colouring that is a (weak) sense of direction using the *smallest* number of colours.

The number of vertices n in a graph G is a trivial upper bound for the number of colours, and the maximum degree Δ_G is a trivial lower bound. However, Δ_G was essentially the *only* known lower bound; the difficulty of proving that $\Delta_G + 2$ colours were necessary for some graph prompted for the conjecture that $\Delta_G + 1$ colours were always sufficient [6]; the conjecture is of course of particular interest for regular graphs. Recently the authors proved that there are graphs requiring $\omega(n/\log n)$ additional colours [2], but the proof uses intensively *random graphs* of high degree: therefore, an extension of the proof to regular graphs appears difficult (as the theory of random regular graphs mainly considers fixed or slowly growing degrees).

In this paper, using a recent result in graph asymptotic enumeration [8], we bypass this problem and show that $\Omega(n \log\log n/\log n)$ additional colours are necessary to give weak sense of direction to all regular graphs. This result strongly disproves the original conjecture, even when restricted to regular graphs. We also show

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that if the overall number of colours used is dependent on the degree d_G it must be $\Omega\left(d_G\sqrt{\log\log d_G}\right)$.

We remark that even if the main proof of this paper is a rather straightforward counting argument, it is based on an asymptotic estimate of the number of regular graphs enjoying a suitable property, and this estimate requires rather involved computations.

A consequence of our proof is that almost all regular graphs in a certain range of degrees (see Theorem 3) have diameter two. There are presently no published results of this kind in the literature (see [10]), although Krivelevitch, Sudakov, Vu and Wormald are preparing a paper on these issues that covers a wider degree range [9]. However, we believe that the techniques used in the proof can be fruitfully applied to many other properties of random regular graphs.

2 Definitions

A *(directed) graph* G is given by a set $V = [n] = \{0, 1, \ldots, n-1\}$ of n vertices and a set $A \subseteq V \times V$ of arcs (note that the graphs in this paper are *not* considered up to isomorphism—using a common terminology, they are *labelled*). We write $P[x, y] \subseteq A^*$ for the set of paths from vertex x to vertex y. A graph is *symmetric* if $\langle y, x\rangle$ is an arc whenever $\langle x, y\rangle$ is.

In this paper we shall always manipulate symmetric loopless directed graphs, which are really nothing but undirected simple graphs (an edge is identified with a pair of opposite arcs). However, the directed symmetric representation allows us to handle more easily the notion of weak sense of direction and the related proofs. In turn, when using asymptotic enumeration results we shall confuse a symmetric loopless directed graph with its undirected simple counterpart.

The (average) degree d_G of a graph G is $|A|/|V|$ (or, in the undirected interpretation, twice the number of edges divided by the number of vertices). Of course, if G is regular (i.e., all vertices have the same number of incoming and outgoing arcs) then d_G is the (in- and out-)degree of every vertex, and one says that G is d_G-regular.

A *colouring* of a graph G is a function $\lambda : A \to \mathscr{L}$, where $\mathscr{L}$ is a finite set of colours; the map $\lambda^* : A^* \to \mathscr{L}^*$ is defined by $\lambda^*(a_1a_2\cdots a_p) = \lambda(a_1)\lambda(a_2)\cdots\lambda(a_p)$. We write $\mathscr{L}_x = \{\lambda(\langle x, y\rangle) \mid \langle x, y\rangle \in A\}$ for the set of colours that x assigns to its outgoing arcs.

Given a graph G coloured by λ, let

$$L = \bigcup_{\langle x,y\rangle\in V^2} \{\lambda^*(\pi) \mid \pi \in P[x, y]\};$$

be the set of all strings that colour paths of G.

A *local naming* for G is a family of injective functions $\beta = \{\beta_x : V \to \mathscr{S}\}_{x\in V}$, with $\mathscr{S}$ a finite set, called the *name space*. Intuitively, each vertex x of G gives to each other vertex y a name $\beta_x(y)$ taken from the name space.

Given a coloured graph endowed with a local naming, a function $f : L \to \mathscr{S}$ is a *coding function* iff

$$\forall x, y \in V \quad \forall \pi \in P[x, y] \qquad f(\lambda^*(\pi)) = \beta_x(y).$$

A coding function translates the colouring of the path along which two vertices x, y are connected into the name that x gives to y. A colouring λ is a *weak sense of direction* for a graph G iff for some local naming there is a coding function[1]. We shall also say that a coloured graph *has* weak sense of direction, or that λ *gives* weak sense of direction to G.

3 Representing Regular Graphs Using Weak Sense of Direction

A coding function f *represents compactly a great deal of information about a graph*, because f tells whether two paths with the same source have the same target. For instance, suppose that we want to exploit (naively) this property to code compactly a (strongly) connected regular graph G with weak of sense of direction. Assume without loss of generality that $\beta_0(x) = x$ for all vertices x, that is, vertex 0 locally gives to all other vertices their "real names". To code G, first specify for each vertex the set of colours of outgoing arcs. Then, give the values of f on every string of colours having length at most $D + 1$, where D is the diameter of G.

To rebuild G from the above data, we proceed as follows: first of all we compute the targets of the arcs out of 0 using f on strings of length one, thus obtaining the set of coloured paths of length one going out of 0. Then, since we know the colours of the arcs going out of the targets of such paths, we can build the set of coloured paths of length two out of 0, and compute their targets using f on strings of length two, and so on. Thus, we will eventually discover all arcs of G, using just the values of f on paths of length $D+1$ at most. Unfortunately this naive attempt is too rough, even for $D = 2$, so we shall use a slightly more sophisticated approach.

Let $\mathscr{C}(n, k)$ be the class of all symmetric k-regular graphs with n vertices that enjoy the following property, which we shall call *property* $A_{3/2}$:

> If x_1, x_2, x_3 are three distinct vertices such that x_2 and x_3 are adjacent, and x_1 is not adjacent to x_2 and not adjacent to x_3, then there exists a vertex $z \notin \{x_1, x_2, x_3\}$ that is adjacent to x_1, x_2 and x_3.

Note that property $A_{3/2}$ is weaker than property A_3 of [2]; as we shall see, for suitable d almost all d-regular graphs enjoy property $A_{3/2}$, and nonetheless graphs satisfying $A_{3/2}$ can be coded compactly. Intuitively for regular graphs $A_{3/2}$ is a connectivity property slightly stronger than having diameter two, since given any pair of vertices x, y we can choose a vertex z adjacent to y (the existence of z is ensured by regularity) and apply $A_{3/2}$, getting a vertex that, in particular, is adjacent both to x and to y. The same argument shows also that a regular graph satisfying $A_{3/2}$ is connected.

Lemma 1. *Let G be a graph satisfying $A_{3/2}$. Let λ be a sense of direction for G with name space $\mathscr{S}$, coding function f and local naming β. Assume without loss of generality that $\mathscr{S} \supseteq [n]$ and $\beta_0(x) = x$ for all vertices x. Then, $\langle x, y\rangle$ is an arc iff one of the following holds:*

[1] In [5] a slightly different definition is given, in which the empty string is not part of L. The results of this paper are not affected by this difference.

– *$x = 0$ and $y = f(a)$ for some colour $a \in \mathscr{L}_0$;*
– *$x = f(a)$ and $y = f(ab)$ for some $a \in \mathscr{L}_0$ and $b \in \mathscr{L}_{f(a)}$;*
– *$x = f(ab)$ and $y = f(ac)$ for some $a \in \mathscr{L}_0$, $b, c \in \mathscr{L}_{f(a)}$, provided that there exists $d \in \mathscr{L}_{f(ab)}$ such that $f(bd) = f(c)$.*

Proof. If a colours an arc going out of 0, the target of the arc is $f(a)$. Moreover, if there is an arc with colour b going out of $f(a)$, the target of such arc is $f(ab)$. For the third case, ab and ac colour paths going out of 0, whereas bd and c colour paths going out of $f(a)$. Since $f(bd) = f(c)$, the latter paths must have the same target; hence the path from 0 coloured ac has the same target as the path coloured abd. Therefore, there must be an arc, coloured d, from $f(ab)$ to $f(ac)$, as in Fig. 1.

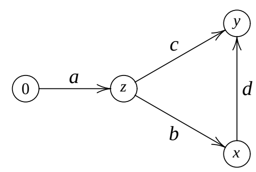

Figure 1. Property $A_{3/2}$ in action.

For the other side of the implication, consider an arc $\langle x, y\rangle$ of G. We have three cases:

– if $x = 0$, then it must correspond to an arc of the form $\langle 0, f(a)\rangle$ for some $a \in \mathscr{L}_0$;
– if x is an outneighbour of 0, then $x = f(a)$ for some $a \in \mathscr{L}_0$, and the arc corresponds to an arc of the form $\langle f(a), f(ab)\rangle$ for some $b \in \mathscr{L}_{f(a)}$;
– finally, assume that $x, y \neq 0$ and that moreover x and y are not outneighbours of 0. By property $A_{3/2}$, there exists a vertex z adjacent to x, y and 0. Let a be the colour of the arc going from 0 to z, b be the colour of the arc going from z to x, c be the colour of the arc going from z to y, and d be the colour of the arc from x to y. We have $f(ab) = \beta_0(x) = x$, $f(ac) = \beta_0(y) = y$ and $f(bd) = \beta_z(y) = f(c)$ (see Fig. 1). □

Following the line of [2], we can use Lemma 1 to code compactly regular graphs as follows:

Theorem 1. *Let $c = c(G) \in \mathbf{N}$ be such that every k-regular graph G with n vertices can be given weak sense direction using no more than $c(G)$ colours. Then every graph in $\mathscr{C}(n, k)$ can be described[2] using $O\big(cn + c^2 \log n\big)$ bits.*

Proof. Let $G \in \mathscr{C}(n, k)$ have weak sense of direction with colouring λ, name space $\mathscr{S}$, local naming β and coding function f. Assume without loss of generality that $\mathscr{S} \supseteq [n]$ and $\beta_0(x) = x$ for every vertex x. Describe G as follows:

[2] From now on, we shall sometimes omit the explicit dependence of functions from their argument, when the latter is clear from the context, thus writing c instead of $c(G)$, d instead of $d(n)$ and so on.

1. give the number of colours c;
2. for every vertex x, use c bits to describe the set $\mathscr{L}_x$;
3. give the values of f on every string of length one or two.

The first data require $\lceil \log c \rceil$ bits, the second one cn bits and the third one $\left(c + c^2\right) \lceil 2 \log n \rceil$ (as we mentioned, $\lceil 2 \log n \rceil$ bits are sufficient to specify a name). From the above description, G can be recovered using Lemma 1. □

4 A Result about Graph Enumeration

The inspiration for this paper came out of a recent breakthrough by McKay and Wormald in asymptotic graph enumeration:

Theorem 2 ([8]). *Let $d = d(n)$ and $\delta_j = \delta_j(n)$, $0 \leq j < n$, be such that* $\min\{d, n - d - 1\} > cn/\log n$ *for some $c > \frac{2}{3}$, $\sum_{j=0}^{n-1} \delta_j = 0$, $\delta_j = O(1)$ uniformly over j, $d + \delta_j$ is an integer for $0 \leq j < n$ and dn is an even integer. Then the number of graphs with n vertices and local degrees $d + \delta_0$, $d + \delta_1, \ldots, d + \delta_{n-1}$ is asymptotic to $\alpha M(n, d)$, where*

$$M(n, d) = \left[2\pi n \left(\frac{d}{n-1} \right)^{d+1} \left(1 - \frac{d}{n-1} \right)^{n-d} \right]^{-n/2}$$

and $\gamma_1/n^{\varepsilon} \leq \alpha \leq \gamma_2$ for suitable positive constants ε, γ_1 and γ_2.

In other words, under the given hypotheses the order of magnitude depends essentially on the average degree only, and not on the specific degrees (but note that α in general will depend on n, on d and on the δ_j's). The original result of McKay and Wormald is much more powerful, as it provides a precise asymptotic estimate for much more varied δ_j's, but the simplification above is sufficient for our purposes.

The above theorem has the following consequence, whose (complex) proof is deferred to the last section:

Theorem 3. *Let $d = o(n)$ satisfy the hypotheses of Theorem 2. Then almost all d-regular graphs satisfy $A_{3/2}$.*

The statement "almost all d-regular graphs satisfy P" means that the number of d-regular graph of order n enjoying P divided by the number of all d-regular graphs of order n goes to 1 as $n \to \infty$. Equivalently, if we consider the standard model of d-regular random graphs [3] in which all d-regular graphs of order n are equiprobable, we can say that the probability that a random graph satisfies P goes to 1 as $n \to \infty$.

Since under the given hypotheses the number of all d-regular graphs is asymptotic to the number of d-regular graphs satisfying $A_{3/2}$, we can use Theorem 2 to get an asymptotic estimate of the size of the class $\mathscr{C}(n, d)$, and thus a lower bound on the number of bits that are necessary to describe a graph belonging to it.

Theorem 4. *Let $d = o(n)$ satisfy the hypotheses of Theorem 2. Then, the number of bits required to describe a graph in $\mathscr{C}(n, d)$ is $\Theta(nd \log(n/d))$.*

Proof. Since by Theorem 3 the number of graphs in $\mathscr{C}(n, d)$ is asymptotic to the number of d-regular graphs, Theorem 2 tells us that the number of bits required is asymptotic to $\log[\alpha M(n, d)]$. If we expand the latter expression killing all terms that are $O(nd)$ we obtain

$$\begin{aligned}\log[\alpha M(n, d)] &= -\frac{n}{2}(1+d)\log\frac{d}{n-1} - \frac{n}{2}(n-d)\log\left(1-\frac{d}{n-1}\right) + O(nd)\\ &= \Theta\left(nd\log\frac{n}{d}\right) + \frac{n}{2}(n-d)\frac{d}{n-1} + O(nd) = \Theta\left(nd\log\frac{n}{d}\right). \square\end{aligned}$$

5 The Main Theorem

We finally put together the upper and lower bounds we obtained:

Theorem 5. *If $g(n) = o(n \log\log n/\log n)$, it is impossible to give (weak) sense of direction to all regular graphs using $d_G + g(n)$ colours. Moreover, it is impossible to give (weak) sense of direction to all regular graphs using $o\big(d_G\sqrt{\log\log d_G}\big)$ colours.*[3]

Proof. If $g = O(n/\log n)$, take any $d = \Theta(n/\log n)$ satisfying the hypotheses of Theorem 2 and note that by Theorem 1 $O\big(n^2/\log n\big)$ bits would be sufficient to describe a graph in $\mathscr{C}(n, d)$, but by Theorem 4 $\Theta\big(n^2\log\log n/\log n\big)$ are required. Otherwise, we can write $g = n\, f(n)/\log n$, with $f(n) = o(\log\log n)$, and take $d = g$. In this case $O\big(n^2 f(n)^2/\log n\big) = o\big(n^2 f(n)\log\log n/\log n\big)$ bits would be sufficient, but $\Theta\big(n^2 f(n)\log\log n/\log n\big)$ are required.

Finally, if $h(m) = o\big(m\sqrt{\log\log m}\big)$ as $m \to \infty$ take any $d = \Theta(n/\log n)$; in this case $O\big(h(d)^2/\log n\big) = o\big(n^2\log\log n/\log n\big)$ bits would be sufficient, but again $\Theta\big(n^2\log\log n/\log n\big)$ are necessary. $\square$

6 A Proof of Theorem 3—Part I

To prove that almost all d-regular graphs satisfy $A_{3/2}$, we show that almost no d-regular graph satisfies $\neg A_{3/2}$. The interesting feature of a d-regular graph G with n vertices that does not satisfy $A_{3/2}$ is that it has a rather precise structure, displayed in Fig. 2, where $A_{3/2}$ does not work on x_1, x_2 and x_3 (in Fig. 2 we draw only edges incident on x_1, x_2 and x_3).

The vertices of G are partitioned into seven sets, depending on their adjacency relations with the three vertices on which $A_{3/2}$ does not work. If we strip x_1, x_2 and x_3 we obtain a new "stripped graph" with $n-3$ vertices and a rather precise degree assignment: clearly all vertices in $V_\varnothing$ will have degree d, all vertices in the sets V_i will have degree $d-1$ and all vertices in the sets V_{ij} will have degree $d-2$. The key point is that such a degree structure still falls under the scope of Theorem 2. Since, as we will show, the average degree d' of a stripped graph is independent of the actual cardinalities of the

[3] That is, for every function $h(m) = o\big(m\sqrt{\log\log m}\big)$ there is a graph G such that $h(d_G)$ colours are not sufficient.

V's, we may hope to bound the number of counterexamples to $A_{3/2}$ using $M(n-3, d')$ to bound carefully the number of stripped graphs. To this goal, we work backwards and define a suitable kind of graph that can be enriched with three vertices so to obtain a d-regular counterexample to $A_{3/2}$ of order n.

An (n, d)-*stripped graph* is a graph S with $n-3$ vertices, endowed with a vertex-colouring function $\pi : [n-3] \to 2^{\{1,2,3\}}$, and satisfying the following conditions: let us write V_1 for the set of vertices coloured by $\{1\}$, V_{12} for the set coloured by $\{1, 2\}$ and so on (formally, $V_X = \pi^{-1}(X)$ for $X \subseteq \{1, 2, 3\}$), and finally let $v_X = |V_X|$; we require that

$$\begin{aligned} \deg(x) + |\pi(x)| &= d \text{ for every vertex } x \text{ of } S \\ v_{123} &= 0 \\ v_1 + v_{12} + v_{13} &= d \\ v_2 + v_{12} + v_{23} &= d - 1 \\ v_3 + v_{13} + v_{23} &= d - 1 \end{aligned}$$

The rationale behind the previous equalities is immediate, looking at Fig. 2. Note that, as a consequence,

$$v_1 + v_2 + v_3 + 2(v_{12} + v_{13} + v_{23}) = 3d - 2.$$

We can associate to each (n, d)-stripped graph S a d-regular counterexample to $A_{3/2}$

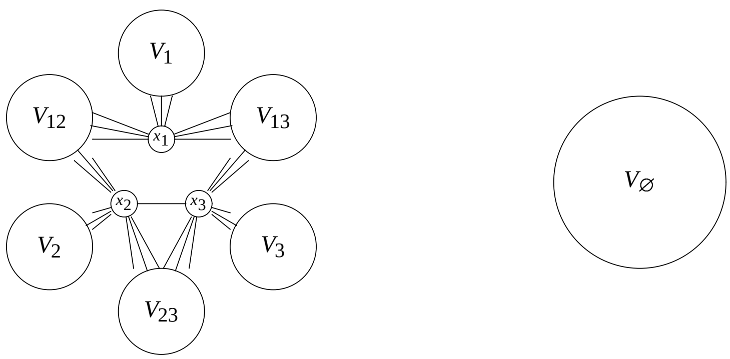

Figure 2. A generic counterexample to $A_{3/2}$.

with n vertices in the following manner: we add three new vertices $n-3$, $n-2$ and $n-1$ to S, and connect vertex $n-y$ to vertex $x < n-3$ if and only if $y \in \pi(x)$. Moreover, vertex $n-2$ is adjacent to vertex $n-3$. It is straightforward to observe that the graph constructed as above is d-regular (because of the condition $\deg(x) + |\pi(x)| = d$). Moreover, vertices $n-3$, $n-2$ and $n-1$ fail to satisfy property $A_{3/2}$; conversely, every d-regular graph of size n whose last three vertices fail to satisfy $A_{3/2}$ may be obtained from a suitable (n, d)-stripped graph using the above construction. Finally, we remark that we can choose for each counterexample to $A_{3/2}$ a relabelling that exactly exchanges the labels of the (say, lexicographically first) three vertices that break $A_{3/2}$ with those of last three vertices. As a result, we have that at most n^3 counterexamples to $A_{3/2}$ correspond to an (n, d)-stripped graph. Our next goal is thus to bound the number of (n, d)-stripped graphs, and to this purpose we state some simple properties of the variables v that are easily derivable from the linear system above:

Lemma 2. *Let S, π define an (n, d)-stripped graph, $k = n-3-v_\varnothing$ and $s = v_{12}+v_{13}$; then:*

1. *the average degree of S is $d' = d - (3d-2)/(n-3)$;*
2. *the following (in)equalities hold:*

$$v_{12} + v_{13} + v_{23} = 3d - 2 - k \tag{1}$$
$$v_2 = k - 2d + v_{13} + 1 \tag{2}$$
$$v_1 + v_2 = k - d - v_{12} + 1 \tag{3}$$
$$\lceil (3d-2)/2 \rceil \le k \le 3d - 2 \tag{4}$$
$$\max(0, 4d - 2k - 2) \le s \le \min(d, 3d - 2 - k) \tag{5}$$
$$\max(0, 2d - k - 1) \le v_{12} \le \min(s, k - 2d + s + 1). \tag{6}$$

Proof. 1. The average degree of S is:

$$\begin{aligned} d' &= \frac{v_\varnothing d + (v_1 + v_2 + v_3)(d-1) + (v_{12} + v_{13} + v_{23})(d-2)}{n-3} \\ &= d - \frac{v_1 + v_2 + v_3 + 2(v_{12} + v_{13} + v_{23})}{n-3} = d - \frac{3d-2}{n-3}. \end{aligned}$$

2. Equation (1) directly follows from the constraints; hence we have $v_2 = d - v_{12} - v_{23} - 1 = k - 2d + v_{13} + 1$, proving (2). Moreover, $v_1 = d - v_{12} - v_{13}$, hence $v_1 + v_2 = k - d - v_{12} + 1$, which is (3). For proving inequality (4), observe that $k + v_{12} + v_{13} + v_{23} = 3d - 2$ implies $k \le 3d - 2$; moreover, since $2(v_{12} + v_{13} + v_{23}) = 3d - 2 - (v_1 + v_2 + v_3)$, we have $k = (v_1 + v_2 + v_3 + 3d - 2)/2 \ge (3d-2)/2$. For (5), first recall that $v_2 = k - 2d + v_{13} + 1$, and similarly $v_3 = k - 2d + v_{12} + 1$; the nonnegativity constraints on v_2 and v_3 give $v_{12} \ge 2d - 1 - k$ (which is the only nontrivial lower bound for the remaining pair of inequalities) and $v_{13} \ge 2d - 1 - k$, hence the lower bound on $s = v_{12} + v_{13}$. On the other hand, $s = 3d - 2 - k - v_{23} \le 3d - 2 - k$ and also $s = v_{12} + v_{13} = d - v_1 \le d$. Finally, for (6), we have $v_{12} = s - v_{13} \le s$, and moreover, since $v_{13} \ge 2d - k - 1$, $s = v_{12} + v_{13} \ge 2d - k - 1 + v_{12}$, hence the bound $v_{12} \le s + k + 1 - 2d$. ◻

How can we bound the number of (n, d)-stripped graphs? Looking at the linear system above it is clear that once we choose values for k, s and v_{12} within the bounds of Lemma 2 all other v's are uniquely determined, as $v_{13} = s - v_{12}$, $v_{23} = 3d - 2 - k - s$, and the remaining values can always be computed (the system has maximum rank). Thus, the number of vertices to be assigned a certain colour is now fixed: we just have to choose *which* vertices will receive a certain colour. This can be done choosing first k vertices out of $n - 3$ (that is, the set of vertices with degree smaller than d); then choosing the $3d-2-k$ vertices out of k that will have degree $d-2$; among the latter we must first choose the s vertices that belong to $V_{12} \cup V_{13}$, and out of these the v_{12} vertices of V_{12}; then, among the $k-(3d-2-k) = 2k-3d+2$ vertices of degree $d-1$ we must choose the $k-d-v_{12}+1$ vertices in $V_1 \cup V_2$, and finally out of these the $d-s$ vertices of V_1. Once also this choice is fixed, the bound of McKay and Wormald tells us that the number of graphs with the sequence of degrees given by the choices above is at most $\gamma_2\, M(n-3, d')$. All in all, we obtain the following horrendous-looking triple summation:

$$\sum_{k=\lceil(3d-2)/2\rceil}^{3d-2} \sum_{s=\max\{0,4d-2k-2\}}^{\min\{d,3d-2-k\}} \sum_{v_{12}=\max\{0,2d-k-1\}}^{\min\{s,k-2d+s+1\}} \binom{n-3}{k}\binom{k}{3d-2-k} \cdot \\ \cdot \binom{3d-2-k}{s}\binom{s}{v_{12}}\binom{2k-3d+2}{k-d-v_{12}+1}\binom{k-d-v_{12}+1}{d-s}\gamma_2\, M(n-3, d')$$

However, things are not as bad as they may seem: the last factor is independent of all summation indices, and the first three binomials are independent of v_{12}, so they can be moved out accordingly. Finally, applying trinomial revision[4] to the last two binomials we remove a dependence on v_{12}, getting to

$$\gamma_2\, M(n-3, d') \sum_{k=\lceil(3d-2)/2\rceil}^{3d-2} \sum_{s=\max\{0,4d-2k-2\}}^{\min\{d,3d-2-k\}} \binom{n-3}{k}\binom{k}{3d-2-k} \cdot \\ \cdot \binom{3d-2-k}{s}\binom{2k-3d+2}{d-s} \sum_{v_{12}=\max\{0,2d-k-1\}}^{\min\{s,k-2d+s+1\}} \binom{s}{v_{12}}\binom{2k-4d+s+2}{k-2d+s-v_{12}+1}.$$

Since we are interested in an upper bound, we can extend the last summation to $0 \le v_{12} \le k-2d+s+1$ and use Vandermonde convolution[5]. The resulting term is a central binomial coefficient of upper index $2k-4d+2s+2$, and can be bounded with $2^{2k-4d+2s+2}$; the part independent of k and s can be moved out, getting to

$$\gamma_2\, M(n-3, d')\, 4^{1-2d} \sum_{k=\lceil(3d-2)/2\rceil}^{3d-2} \sum_{s=\max\{0,4d-2k-2\}}^{\min\{d,3d-2-k\}} \binom{n-3}{k} \cdot \binom{k}{3d-2-k} \cdot \\ \binom{3d-2-k}{s}\binom{2k-3d+2}{d-s} 4^{k+s}.$$

There is not much more we can do about the summation term. The summation indices k and s appear almost everywhere, so we take a different approach: since the range of summation is extremely small (see Fig. 3) when compared to the summands, we can try to find an upper bound for the latter. To this aim, we study the behaviour of finite differences in k and s over the range of summation. This is a standard technique

[4] The *trinomial revision* theorem states that $\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$—see, e.g., [7].

[5] *Vandermonde convolution*: $\sum_{k=0}^{n} \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$, *ibid.*.

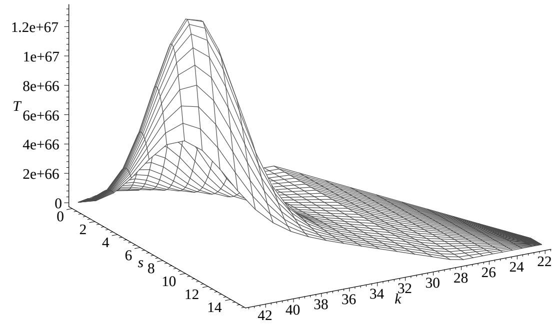

Figure 3. A look at the behaviour of $T_n(s,k)$ for $n = 100$ and $d = \lfloor n/\log n \rfloor$.

used when binomials are involved, as the sign of finite differences usually depends in a simple way on a low-degree polynomial. Indeed, if we let

$$T_n(s,k) = \binom{n-3}{k}\binom{k}{3d-2-k}\binom{3d-2-k}{s}\binom{2k-3d+2}{d-s}4^{k+s}$$

it is immediate to discover that

$$T_n(s,k+1) \geq T_n(s,k) \iff K_n(s,k) \geq 0 \tag{7}$$
$$T_n(s+1,k) \geq T_n(s,k) \iff S_n(s,k) \geq 0, \tag{8}$$

where

$$K_n(s,k) = (4d+6-4n)k - s^2 + (5+8d-4n)s + 12nd - 8n - 8d + 12 - 16d^2$$
$$S_n(s,k) = (2s-2-4d)k + 3s^2 + (-12d+4)s + 12d^2 - 4d - 3.$$

Since both polynomials are linear in k (with ultimately negative coefficient), we can make conditions (7) and (8) explicit, obtaining two rational functions $z_K(s)$ and $z_S(s)$ such that the inequalities

$$k \leq z_K(s) = -\frac{1}{2}\frac{s^2 + (4n-8d-5)s + 8n - 12nd + 8d - 12 + 16d^2}{2n-2d-3}$$
$$k \leq z_S(s) = -\frac{1}{2}\frac{3s^2 + (4-12d)s + 12d^2 - 3 - 4d}{s-1-2d}$$

are equivalent to (7) and (8), respectively.

Armed with the knowledge above, we now try to answer to the following question: which conditions must a pair of integers $\langle s,k\rangle$ satisfy to be a local maximum of T_n? Clearly the strict version of both condition (7) and condition (8) must be false at $\langle s,k\rangle$, for the "next" integers on the plane must feature a smaller or equal value of T_n; similarly, condition (7) must be true at $\langle s,k-1\rangle$ and condition (8) must be true at $\langle s-1,k\rangle$. All in all, we obtain the following set of constraints:

$$k \geq z_K(s) \qquad\qquad k \geq z_S(s)$$
$$k \leq z_K(s) + 1 \qquad\qquad k \leq z_S(s-1)$$

The situation is depicted in Fig. 4 for $n = 10^6$ and $d = \lfloor n/\log n \rfloor$. The thicker curves represent the constraints involving z_K, and the thinner ones the constraints involving z_S. The region satisfying the constraints is the lozenge formed by the four curves (an

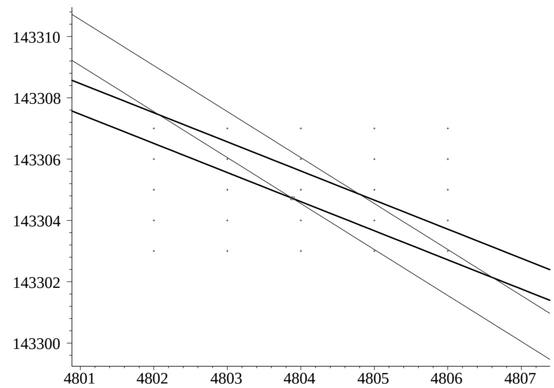

Figure 4. The constraints on the local maxima of T_n

easy check on the values of z_K, z_S and their derivatives on the range of summation shows that indeed this is always the case). The marked point at the intersection of z_S and z_K is the only common zero of K_n and S_n in the range of summation, and can be easily computed with elementary techniques. Its coordinates are

$$\hat{s} = 2\frac{d^2}{n} + O\left(\frac{d^3}{n^2}\right) \qquad\qquad \hat{k} = 3d - 3\frac{d^2}{n} + O\left(\frac{d^3}{n^2}\right). \tag{9}$$

Our goal now is to show that knowing $\hat{k}$ and $\hat{s}$ with the precision shown above is sufficient to know with the same precision the location of the global maximum of T_n over the pairs of integers in the summation range. In other words, we just have to show that that all integral points in the lozenge are not too far from $\langle \hat{s}, \hat{k} \rangle$. To this purpose, it is sufficient to give a rough estimate of the size of a rectangle containing the lozenge, for instance the rectangle defined by the upper and lower intersection points, which happen to be also the leftmost and rightmost, respectively. Theoretically it is possible to compute this points exactly, but unfortunately they are the unmanageable roots of two cubic equations. However, standard algebraic manipulation shows that

$$z_S\left(\hat{s} - \gamma\frac{d^3}{n^2}\right) - z_K\left(\hat{s} - \gamma\frac{d^3}{n^2}\right) - 1 = \frac{1}{2}(\gamma - 2)\frac{d^3}{n^2} - r + o\left(\frac{d^3}{n^2}\right)$$
$$z_K\left(\hat{s} + \gamma\frac{d^3}{n^2}\right) - z_S\left(\hat{s} + \gamma\frac{d^3}{n^2} - 1\right) = \frac{1}{2}(\gamma + 2)\frac{d^3}{n^2} + r + o\left(\frac{d^3}{n^2}\right),$$

where $r = \hat{s} - 2d^2/n = O(d^3/n^3)$. In other words, we can choose a fixed γ so that ultimately at distance $\gamma d^3/n^2$ to the left of $\hat{s}$ the thinner curves are both over the thicker ones, and conversely at distance $\gamma d^3/n^2$ to the right. This shows that the width of the lozenge is $O(d^3/n^2)$. Finally, it is easy to check that $z_S(\hat{s} - \gamma d^3/n^2) - z_K(\hat{s} + \gamma d^3/n^2) = O(d^3/n^2)$, so the lozenge is included in a rectangle whose sides are both $O(d^3/n^2)$. We conclude the global maximum on integers pair is attained at a point $\langle \bar{s}, \bar{k}\rangle$ satisfying

$$\bar{s} = 2\frac{d^2}{n} + O\left(\frac{d^3}{n^2}\right) \qquad\qquad \bar{k} = 3d - 3\frac{d^2}{n} + O\left(\frac{d^3}{n^2}\right),$$

that is, modulo error terms $O(d^3/n^2)$, the same as (9). As the reader will see, this will be sufficient for our purposes.

We have thus finally reached our goal: our bound on the number of (n, d)-stripped graphs becomes

$$\gamma_2\, M(n-3, d')\, d\lfloor(3d-2)/2\rfloor T_n(\bar{s}, \bar{k}),$$

and the proof that

$$\frac{n^3\gamma_2\, M(n-3, d')\, d\lfloor(3d-2)/2\rfloor T_n(\bar{s}, \bar{k})}{(\gamma_1/n^{\varepsilon})M(n, d)} \to 0$$

is now amenable to standard asymptotic techniques. In the next section we provide a full (and rather tedious) proof.

7 A Proof of Theorem 3—Part II

We start with a few obvious considerations: since we have exponentials around, we need to estimate the natural logarithm of our expression. In doing so, we plan in advance to consider only summands that are of order at least d^3/n^2: in particular, the factor $\gamma_2 n^{3+\varepsilon}\lfloor(3d-2)/2\rfloor/\gamma_1$ plays no rôle. We remark the following asymptotic relations between the principal quantities we will have to manage:

$$d = \Omega\left(\frac{n}{\log n}\right) \quad d \sim d' = o(n) \quad d - d' = \frac{3d}{n} + O\left(\frac{1}{n}\right) \quad \frac{d^{t+1}}{n^t} = o\left(\frac{d^t}{n^{t-1}}\right)$$

We start by computing the natural logarithm of $M(n-3,d')/M(n,d)$:

$$\begin{aligned}
\ln\frac{M(n-3,d')}{M(n,d)} &= -\frac{n-3}{2}[\ln(2\pi)+\ln(n-3)-(n-2)\ln(n-4)+\\
&\quad +(d'+1)\ln d' + (n-3-d')\ln(n-4-d')]+\\
&\quad +\frac{n}{2}[\ln(2\pi)+\ln n-(n+1)\ln(n-1)+(d+1)\ln d+(n-d)\ln(n-1-d)]\\
&= -\frac{n}{2}\bigg[\ln\frac{n-3}{n} - n\ln\frac{n-4}{n-1} + \ln\frac{(n-4)^2(n-1)}{(n-4-d')^3} + (d+1)\ln\frac{d'}{d} - \frac{3d-2}{n-3}\ln d' +\\
&\quad + (n-d)\ln\frac{n-4-d'}{n-1-d} + \frac{3d-2}{n-3}\ln(n-4-d')\bigg] + \frac{3}{2}[\ln(2\pi)+\\
&\quad + \ln(n-3)-(n-2)\ln(n-4)+(d'+1)\ln d' + (n-3-d')\ln(n-4-d')]
\end{aligned}$$

Since we are going to expand asymptotically all logarithms, we notice that

$$\frac{(n-4)^2(n-1)}{(n-4-d')^3} = 1+3\frac{d'}{n}+6\frac{d'^2}{n^2}+10\frac{d'^3}{n^3}+O\left(\frac{d'^4}{n^4}\right),$$

so we obtain

$$\begin{aligned}
&= -\frac{n}{2}\bigg[\ln\left(1-\frac{3}{n}\right) - n\ln\left(1-\frac{3}{n-1}\right) + \ln\left(1+3\frac{d'}{n}+6\frac{d'^2}{n^2}+10\frac{d'^3}{n^3}+O\left(\frac{d'^4}{n^4}\right)\right)+\\
&\quad +(d+1)\ln\left(1-\frac{3d-2}{d(n-3)}\right) - \frac{3d-2}{n-3}\ln d' + (n-d)\ln\left(1+\frac{d-d'-3}{n-1-d}\right)+\\
&\quad +\frac{3d-2}{n-3}\ln(n-4-d')\bigg] + \frac{3}{2}[\ln(2\pi)+\ln(n-3)-(n-2)\ln(n-4)+\\
&\quad +(d'+1)\ln d' + (n-3-d')\ln(n-4-d')].
\end{aligned}$$

Note that since $\ln(1+x) = x - x^2/2 + x^3/3 + O(x^4)$ for $x \to 0$, if $g = o(f)$ then

$$\begin{aligned}
\ln(f+g) &= \ln f + \frac{g}{f} - \frac{g^2}{2f^2} + \frac{g^3}{3f^3} + O\left(\frac{g^4}{f^4}\right)\\
(f+g)\ln(f+g) &= (f+g)\ln f + g + \frac{g^2}{2f} - \frac{g^3}{6f^2} + O\left(\frac{g^4}{f^3}\right).
\end{aligned}$$

Applying the expansions above and systematically killing all terms that are $o\left(d^3/n^2\right)$ we obtain

$$\begin{aligned}
&\ln\frac{M(n-3,d')}{M(n,d)} \\
&= -\frac{n}{2}\Bigg[\frac{3n}{n-1} + 3\frac{d'}{n} + 6\frac{d'^2}{n^2} + 10\frac{d'^3}{n^3} - \frac{9}{2}\frac{d'^2}{n^2} - 18\frac{d'^3}{n^3} + 9\frac{d'^3}{n^3} - \frac{(d+1)(3d-2)}{d(n-3)} + \\
&\quad -\frac{3d-2}{n-3}\ln d' + \frac{(n-d)(d-d'-3)}{n-1-d} + \frac{3d-2}{n-3}\ln n + \\
&\quad + \frac{3d-2}{n-3}\left(-\frac{4+d'}{n} - \frac{1}{2}\frac{(4+d')^2}{n^2}\right)\Bigg] + \frac{3}{2}\Bigg[-n\ln n + d'\ln d' + (n-d')\ln n + \\
&\quad + (n-d')\left(-\frac{4+d'}{n} - \frac{1}{2}\frac{(4+d')^2}{n^2} - \frac{1}{3}\frac{(4+d')^3}{n^3}\right)\Bigg] + o\left(\frac{d^3}{n^2}\right) \\
&= -3d + \frac{3}{2}\frac{d^2}{n} + \frac{1}{2}\frac{d^3}{n^2} + 3d\ln\frac{d}{n} + o\left(\frac{d^3}{n^2}\right),
\end{aligned}$$

where the last passage is just algebra, once one notes that it is possible to replace d' with d inside logarithms (the resulting error is within our bound).

We now approach the rest of the limit. We want to estimate the behaviour of

$$\begin{aligned}
T_n(\bar{s},\bar{k}) &= \binom{n-3}{\bar{k}}\binom{\bar{k}}{3d-2-\bar{k}}\binom{3d-2-\bar{k}}{s}\binom{2\bar{k}-3d+2}{d-\bar{s}}4^{\bar{k}+\bar{s}} \\
&= \frac{4^{\bar{k}+\bar{s}}(n-3)!}{\bar{s}!(n-3-\bar{k})!(3d-2-\bar{k}-\bar{s})!(d-\bar{s})!(2\bar{k}-4d+2+\bar{s})!}
\end{aligned}$$

using the following asymptotic identity derived from Stirling approximation:

$$\ln[(f+g)!] = (f+g)\ln f - f + \frac{g^2}{2f} - \frac{g^3}{6f^2} + O\left(\frac{g^4}{f^3}\right) + O(\ln(f+g)),$$

which is true when $g = o(f)$, and always keeping in mind that

$$\begin{aligned}
\bar{k} &= 3d - 3\frac{d^2}{n} + O\left(\frac{d^3}{n^2}\right) & \bar{s} &= 2\frac{d^2}{n} + O\left(\frac{d^3}{n^2}\right) \\
3d - \bar{k} - \bar{s} &= \frac{d^2}{n} + O\left(\frac{d^3}{n^2}\right) & 2\bar{k} - 4d + \bar{s} &= 2d - 4\frac{d^2}{n} + O\left(\frac{d^3}{n^2}\right).
\end{aligned}$$

Applying systematically the identities above we obtain:

$$\begin{aligned}
\ln T_n\left(\bar{k},\bar{s}\right) &= (\bar{k}+\bar{s}-2d+1)\ln 4 + (n-3)\ln n - n - (n-\bar{k}-3)\ln n + n + \\
&\quad -\frac{(\bar{k}+3)^2}{2n} - \frac{(\bar{k}+3)^3}{6n^2} - \bar{s}\ln\frac{2d^2}{n} + 2\frac{d^2}{n} - (3d-2-\bar{k}-\bar{s})\ln\frac{d^2}{n} + \frac{d^2}{n} + \\
&\quad -(d-\bar{s})\ln d + d - \frac{\bar{s}^2}{2d} - (2\bar{k}-4d+2+\bar{s})\ln(2d) + 2d + \\
&\quad -\frac{1}{4d}\left[-4\frac{d^2}{n} + O\left(\frac{d^3}{n^2}\right)\right]^2 + o\left(\frac{d^3}{n^2}\right) \\
&= (\bar{k}+\bar{s}-2d)\ln 4 + \bar{k}\ln n - \frac{(\bar{k}+3)^2}{2n} - \frac{(\bar{k}+3)^3}{6n^2} - \bar{s}\ln\frac{2d^2}{n} + 3\frac{d^2}{n} + \\
&\quad -(3d-\bar{k}-\bar{s})\ln\frac{d^2}{n} - (d-\bar{s})\ln d - \frac{\bar{s}^2}{2d} - (2\bar{k}-4d+\bar{s})\ln(2d) + \\
&\quad + 3d - 4\frac{d^3}{n^2} + o\left(\frac{d^3}{n^2}\right) = 3d - 3d\ln\frac{d}{n} - \frac{3}{2}\frac{d^2}{n} - \frac{3}{2}\frac{d^3}{n^2} + o\left(\frac{d^3}{n^2}\right),
\end{aligned}$$

where again the last passage is just algebra (all logarithms cancel out happily). Finally, we put together everything, getting to

$$-3d + \frac{3}{2}\frac{d^2}{n} + \frac{1}{2}\frac{d^3}{n^2} + 3d\ln\frac{d}{n} + 3d - 3d\ln\frac{d}{n} - \frac{3}{2}\frac{d^2}{n} - \frac{3}{2}\frac{d^3}{n^2} + o\left(\frac{d^3}{n^2}\right) \to -\infty,$$

as required.

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